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Classification of solutions to the reflection equation for two-component systems

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Abstract. The symmetries, especially those related to the R-transformation, of the reflection equation (RE) for two-component systems are analysed. The classification of solutions to the RE for eight-, six- and seven-vertex-type R-matrices is given. All solutions can be obtained from those corresponding to the standard R-matrices by K-transformation. For the free-fermion models, the boundary matrices have property tr $K_+(0) = 0$, and the free-fermion-type R-matrix with the same symmetry as that of a Baxter type corresponds to the same form of a K_--matrix for the Baxter type. We present the Hamiltonians for the open spin systems connected with our solutions. In particular, the boundary Hamiltonian of seven-vertex models is obtained with a generalization to the Sklyanin formalism.

1. Introduction

In the framework of the quantum inverse scattering method (QISM) [1–5], the Yang–Baxter equation (YBE)

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{1.1}$$

is a sufficient condition for the integrability of systems with a periodic boundary condition (BC). Given an R-matrix solution to YBE (1.1), we can construct the Lax operator of certain models at a suitable representation of R, and hence transfer matrix $t(u)$. The YBE ensures that $t(u)$ commutes with each other for different spectrum parameters. So, if we expand $t(u)$ with respect to the spectrum parameter u, the coefficients are a set of conserved quantities which satisfy Liouville's criterion of integrability [6, 7].

However, when considering systems on a finite interval with independent boundary conditions at each end, we have to introduce reflection matrices $K_\pm(u)$ to describe such boundary conditions. Sklyanin assumed that the R-matrix has the following symmetries [8]:

regularity: $R(0) \propto P$;
P-symmetry: $P_{12}R_{12}(u)P_{12} = R_{21}(u) = R_{12}(u)$;
T-symmetry: $R_{12}^{t_1t_2} = R_{12}(u)$;
unitarity: $R_{12}(u)R_{21}(-u) \propto \text{id}$;
crossing unitarity: $R_{12}^{t_1}(u)R_{21}^{t_1}(-u-2\eta) \propto \text{id}$,

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where η is the crossing parameter and P_{12} is the permutation matrix. t_1, t_2 denote transpositions in space, V_1 and V_2, respectively. In order that the BC are compatible with integrability, the reflection matrices should obey so-called reflection equations (RE), or boundary Yang–Baxter equations (BYBE) [8–10]

$$R_{12}(u-v) \overset{1}{K}_-(u) R_{21}(u+v) \overset{2}{K}_-(v) = \overset{2}{K}_-(v) R_{12}(u+v) \overset{1}{K}_-(u) R_{21}(u-v), \tag{1.2}$$

$$\begin{aligned} &R_{12}(-u+v) \overset{1}{K}{}^{t_1}_+(u) R_{12}(-u-v-2\eta) \overset{2}{K}{}^{t_2}_+(v) \\ &\qquad = \overset{2}{K}{}^{t_2}_+(v) R_{12}(-u-v-2\eta) \overset{1}{K}{}^{t_1}_+(u) R_{12}(-u+v) \end{aligned} \tag{1.3}$$

where $\overset{1}{K}_\pm = K_\pm \otimes 1$, $\overset{2}{K}_\pm = 1 \otimes K_\pm$, and $R(u)$ satisfies YBE (1.1). So for a solution $K_-(u)$ to RE (1.2), the relation

$$K_+(u) = K^t_-(-u-\eta) \tag{1.4}$$

gives the solution to equation (1.3). Nevertheless, not all R-matrices possess the above-mentioned properties, some generalizations should be made (see e.g. [11]). As we will see in section 4, the seven-vertex (7V) models are also beyond Sklyanin's formalism, for their R-matrices do not enjoy T-symmetry. It was stated in [12] that if an R-matrix has regularity, unitarity and crossing unitarity symmetries, but does not have PT-invariance, we can propose $K_+(u)$ to satisfy the equation:

$$\begin{aligned} &R_{12}(-u+v) \overset{1}{K}_+(u) R_{21}(-u-v-2\eta) \overset{2}{K}_+(v) \\ &\qquad = \overset{2}{K}_+(v) R_{12}(-u-v-2\eta) \overset{1}{K}_+(u) R_{12}(-u+v) \end{aligned} \tag{1.5}$$

and the integrability can be proved as well. There is also a relation between the solutions of equations (1.2) and (1.5)

$$K_+(u) = K_-(-u-\eta). \tag{1.6}$$

We will find later that the Baxter type and free-fermion type-I solutions of 7V models are in this case.

Due to the significance of the RE, a lot of work has been directed to the study of their solutions [11, 13–18], and the Hamiltonians of the systems with such boundary conditions are also constructed. However, most of those works are based on the R-matrices which are derived directly from the parametrization of the statistical weight in vertex models. There in fact exist many kinds of R-matrices according to the classification of eight-vertex (8V) and six-vertex(6V)-type solutions of both the YBE and the coloured YBE [19–21]. It is tedious to solve the reflection equation for every R-matrix. Fortunately, all those R-matrices of two-component systems can be obtained by applying particular solution transformation to standard (or gauge) ones [20] which satisfy certain initial conditions (let us call solution transformation of the R-matrix as R-transformation for brevity). The word 'two-component' means that there exist two states in the system: particles and antiparticles in field theory, spin up and down in a spin system, arrow up and down or right and left in a lattice model (see [22]). After a detailed study of the RE, we can show that there exists a corresponding transformation to the K-matrix (we call it K-transformation) to keep RE invariant under R-transformation. Therefore, we only need to concentrate on K-matrices for the standard R-matrices.

In this paper we shall focus our attention on solutions to reflection equations of two-component systems up to K-transformation. The solutions are divided into three cases, each of which corresponds to 8V, 6V and 7V models and will be discussed in sections 2–4, respectively. For each case, we first analyse the symmetries of the RE, especially those related to the R-transformations, and then find solutions to the RE for the Baxter-type and free-fermion type standard R-matrices, respectively. We put emphasis on new solutions, but for completeness, we also give the solutions obtained by others.

In section 5, for those solutions given in previous sections, we shall construct the corresponding local Hamiltonian of the open spin-chain. The local Hamiltonian means that it only consists of nearest-neighbour interaction terms. A system with such a Hamiltonian can be viewed as having coupling with magnetic field on its ends. Finally we shall argue that for all boundary conditions to the free-fermion models, the reflection matrices $K_+(u)$ have property $\operatorname{tr} K_+(0) = 0$. This property requires us to derive the Hamiltonian from the second derivative of the transfer matrix [16]. In section 6, we make some remarks and discussions.

2. K_--matrices to the 8V model

In this section, we shall first study the symmetries of RE and give the K-transformation corresponding to the R-transformation in [20]. With these discussions, we can concentrate our attention on the RE for the standard 8V R-matrices, which are divided into three types: Baxter type (or XYZ spin-chain [5]), free-fermion type I (or XY model [23, 25]), and free-fermion type II. All K_--matrices associated with these R-matrices are given.

2.1. Symmetries of reflection equation

The general 8V R-matrix and the corresponding $K_-(u)$ matrix are expressed in the following forms respectively:

$$R(u) = \begin{pmatrix} \omega_1(u) & 0 & 0 & \omega_7(u) \\ 0 & \omega_2(u) & \omega_5(u) & 0 \\ 0 & \omega_6(u) & \omega_3(u) & 0 \\ \omega_8(u) & 0 & 0 & \omega_4(u) \end{pmatrix} \tag{2.1}$$

$$K_-(u) = \begin{pmatrix} a_1(u) & a_2(u) \\ a_3(u) & a_4(u) \end{pmatrix}. \tag{2.2}$$

Assuming that $R(u)$ is a solution to YBE (1.1), then as studied in [20], there are four symmetries for the 8V-type R-matrix (2.1).

(R.A) *Symmetry of interchanging indices.* If we exchange the elements of $R(u)$ as $\omega_1(u) \leftrightarrow \omega_4(u)$, $\omega_2(u) \leftrightarrow \omega_3(u)$ or $\omega_5(u) \leftrightarrow \omega_6(u)$, $\omega_7(u) \leftrightarrow \omega_8(u)$, then the new matrix also satisfies YBE (1.1).

(R.B) *The scaling symmetry.* Multiplication of $R(u)$ by an arbitrary function $f(u)$ is still a solution to YBE (1.1).

(R.C) *Symmetry of spectral parameter.* If we take a new spectral parameter $\bar{u} = \lambda u$, where λ is a constant complex number, the new matrix $R(\bar{u})$ is still a solution to YBE (1.1).

(R.D) *Symmetry of weight functions.* If we replace weight functions $\omega_7(u)$, $\omega_8(u)$ by the new ones

$$\bar{\omega}_7(u) = s^{-1}\omega_7(u) \qquad \bar{\omega}_8(u) = s\omega_8(u) \tag{2.3}$$

where s is a non-zero complex constant, the new matrix is still a solution to YBE (1.1).

The symmetries (R.A)–(R.D) are called solution transformations (or R-transformations) of an 8V-type solution of YBE (1.1). It is convenient for later discussion to use such notation as follows:

$$\omega_i(u - v) = u_i \qquad \omega_i(u + v) = v_i$$
$$a_i(u) = x_i \qquad a_i(v) = y_i.$$

Substituting the matrices R and K_- into the reflection equation (1.2), we get 16 component equations, which are divided into groups according to symmetries of the indices:

$$\begin{aligned}&(u_7v_8-u_8v_7)x_1y_4-u_8v_2x_2y_2+u_7v_3x_3y_3+u_4v_6(x_2y_3-x_3y_2)=0\\&(u_7v_8-u_8v_7)x_4y_1-u_8v_2x_2y_2+u_7v_3x_3y_3+u_1v_5(x_2y_3-x_3y_2)=0\end{aligned}\tag{A.1}$$

$$\begin{aligned}&(u_2v_3-u_3v_2)x_1y_4+u_2v_8x_2y_2-u_3v_7x_3y_3+u_5v_1(x_3y_2-x_2y_3)=0\\&(u_2v_3-u_3v_2)x_4y_1+u_2v_8x_2y_2-u_3v_7x_3y_3+u_6v_4(x_3y_2-x_2y_3)=0\end{aligned}\tag{A.2}$$

$$\begin{aligned}&u_7v_1x_1y_1-u_7v_4x_4y_4-u_1v_7x_1y_4+u_4v_7x_4y_1+(u_4-u_1)v_2x_2y_2\\&\qquad+u_7(v_5-v_6)x_3y_2=0\\&u_8v_1x_1y_1-u_8v_4x_4y_4-u_1v_8x_1y_4+u_4v_8x_4y_1+(u_4-u_1)v_3x_3y_3\\&\qquad+u_8(v_5-v_6)x_2y_3=0\end{aligned}\tag{A.3}$$

$$\begin{aligned}&u_3v_6x_1y_1-u_3v_5x_4y_4-u_6v_3x_1y_4+u_5v_3x_4y_1+(u_5-u_6)v_8x_2y_2\\&\qquad+u_3(v_4-v_1)x_3y_2=0\\&u_2v_6x_1y_1-u_2v_5x_4y_4-u_6v_2x_1y_4+u_5v_2x_4y_1+(u_5-u_6)v_7x_3y_3\\&\qquad+u_2(v_4-v_1)x_2y_3=0\end{aligned}\tag{A.4}$$

$$\begin{aligned}&(u_1v_1-u_3v_2)x_1y_2+(u_7v_8-u_5v_5)x_4y_2-u_5v_1x_2y_1+u_1v_5x_2y_4-u_3v_7x_3y_1\\&\qquad+u_7v_3x_3y_4=0\\&(u_1v_1-u_2v_3)x_1y_3+(u_8v_7-u_5v_5)x_4y_3-u_5v_1x_3y_1+u_1v_5x_3y_4-u_2v_8x_2y_1\\&\qquad+u_8v_2x_2y_4=0\\&(u_4v_4-u_3v_2)x_4y_2+(u_7v_8-u_6v_6)x_1y_2-u_6v_4x_2y_4+u_4v_6x_2y_1-u_3v_7x_3y_4\\&\qquad+u_7v_3x_3y_1=0\\&(u_4v_4-u_2v_3)x_4y_3+(u_8v_7-u_6v_6)x_1y_3-u_6v_4x_3y_4+u_4v_6x_3y_1-u_2v_8x_2y_4\\&\qquad+u_8v_2x_2y_1=0\end{aligned}\tag{A.5}$$

$$\begin{aligned}&u_6v_2x_1y_2-u_1v_7x_1y_3+u_2v_5x_4y_2-u_7v_4x_4y_3+(u_2v_1-u_1v_2)x_2y_1\\&\qquad+(u_6v_7-u_7v_6)x_3y_1=0\\&u_6v_3x_1y_3-u_1v_8x_1y_2+u_2v_5x_4y_3-u_8v_4x_4y_2+(u_3v_1-u_1v_3)x_3y_1\\&\qquad+(u_6v_8-u_8v_6)x_2y_1=0\\&u_5v_2x_4y_2-u_4v_7x_4y_3+u_2v_6x_1y_2-u_7v_1x_1y_3+(u_2v_4-u_4v_2)x_2y_4\\&\qquad+(u_5v_7-u_7v_5)x_3y_4=0\\&u_5v_3x_4y_3-u_4v_8x_4y_2+u_3v_6x_1y_3-u_8v_1x_1y_2+(u_3v_4-u_4v_3)x_3y_4\\&\qquad+(u_5v_8-u_8v_5)x_2y_4=0.\end{aligned}\tag{A.6}$$

After a careful study of the above equations, we find that if one applies the following transformations to $K_-(u)$ under the transformations (R.A)–(R.D), the system of equations (A) remains invariant:

(K.A) *The symmetry of interchanging indices.* This symmetry will be discussed for each type of R-matrix later.

(K.B) *The scalar symmetry.* If we multiply $K_-(u)$ by an arbitrary function $g(u)$, the new matrix $g(u)K_-(u)$ is still a solution to RE. On the other hand, all the R-matrices up to an arbitrary scalar function have the same reflection matrix.

(K.C) *The symmetry of spectral parameters.* If we take a new spectral parameter $\bar{u} = \lambda u$, where λ is any constant, the new matrix $K(\bar{u})$ also satisfies RE for $R(\bar{u})$.

(K.D) *The symmetry of weight function.* If applying the transformation (R.D) to $R(u)$, we can make a corresponding K-transformation on $K_-(u)$:

$$\bar{a}_3(u) = \sqrt{s} a_3(u) \qquad \bar{a}_2(u) = \sqrt{s}^{-1} a_2(u) \tag{2.4}$$

keeping $a_1(u), a_4(u)$ unchanged. The new $K_-(u)$ matrix is also a solution to RE for the new R-matrix.

Considering the above symmetries for the the R-matrix and K_--matrix, we can focus our attention on the standard R-matrix with the restrictions [20]

$$\omega_5(u) = \omega_6(u) = 1 \qquad \omega_7(u) = \omega_8(u) \tag{2.5}$$

and initial condition

$$R_{12}(0) = P_{12}. \tag{2.6}$$

Note that from condition (2.5), we only need consider R-transformation (R.A) of interchanging indices $\omega_1 \leftrightarrow \omega_4$ and $\omega_2 \leftrightarrow \omega_3$ hereafter. All R-matrices are classified into two classes: Baxter type and free-fermion type, according to whether or not the elements of the R-matrix satisfy the free-fermion condition [19, 23, 24]

$$\omega_1(u)\omega_4(u) + \omega_2(u)\omega_3(u) - \omega_5(u)\omega_6(u) - \omega_7(u)\omega_8(u) = 0. \tag{2.7}$$

The RE corresponding to these two kinds of R-matrices has very different properties. We shall discuss solutions to RE for these gauge R-matrices respectively.

2.2. Baxter type

The gauge R-matrix of a Baxter type was first derived by Baxter [5], and has the following parametrization:

$$\begin{aligned} \omega_1(u) &= \omega_4(u) = \operatorname{sn}(u+h)/\operatorname{sn} h \\ \omega_2(u) &= \omega_3(u) = \operatorname{sn} u/\operatorname{sn} h \\ \omega_5(u) &= \omega_6(u) = 1 \\ \omega_7(u) &= \omega_8(u) = k \operatorname{sn} u \operatorname{sn}(u+h) \end{aligned} \tag{2.8}$$

where $\operatorname{sn} u, \operatorname{cn} u, \operatorname{dn} u$ are Jacobi elliptic functions of modulus k. It is a high-symmetric one with $\omega_1(u) = \omega_4(u)$, $\omega_2(u) = \omega_3(u)$, so the transformation of interchanging indices (R.A) has no effect in this case.

The K_--matrices in this case have been widely discussed in [13–15, 18]. For completeness, we list here the main results. The most general one was given in [15, 18] as follows:

$$K_-(u) = \begin{pmatrix} \nu \operatorname{sn}(\alpha - u) & \mu \operatorname{sn}(2u) \frac{\lambda(1-k\operatorname{sn}^2 u)+1+k\operatorname{sn}^2 u}{1-k^2\operatorname{sn}^2\alpha \operatorname{sn}^2 u} \\ \mu \operatorname{sn}(2u) \frac{\lambda(1-k\operatorname{sn}^2 u)-1-k\operatorname{sn}^2 u}{1-k^2\operatorname{sn}^2\alpha \operatorname{sn}^2 u} & \nu \operatorname{sn}(\alpha + u) \end{pmatrix} \tag{2.9}$$

where $\mu, \nu, \lambda, \alpha$ are free parameters, and the other special solutions can be obtained by setting these parameters to take special values.

2.3. Free-fermion type I

The R-matrix of the free-fermion type I is less symmetric than that of the Baxter type. In this case, $\omega_2(u) = \omega_3(u)$, but $\omega_1(u) \neq \omega_4(u)$. The reflection equation is equivalent to five systems of equations:

$$
\begin{aligned}
&u_2v_7(x_2y_2 - x_3y_3) + u_5v_1(x_3y_2 - x_2y_3) = 0\\
&u_2v_7(x_2y_2 - x_3y_3) + u_5v_4(x_3y_2 - x_2y_3) = 0\\
&u_7v_2(x_2y_2 - x_3y_3) + u_1v_5(x_3y_2 - x_2y_3) = 0\\
&u_7v_2(x_2y_2 - x_3y_3) + u_4v_5(x_3y_2 - x_2y_3) = 0
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
&u_7v_1x_1y_1 - u_1v_7x_1y_4 + u_4v_7x_4y_1 - u_7v_4x_4y_4 + (u_4 - u_1)v_2x_2y_2 = 0\\
&u_7v_1x_1y_1 - u_1v_7x_1y_4 + u_4v_7x_4y_1 - u_7v_4x_4y_4 + (u_4 - u_1)v_2x_3y_3 = 0
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
&u_2v_5(x_1y_1 - x_4y_4) + u_5v_2(x_4y_1 - x_1y_4) + u_2(v_4 - v_1)x_2y_3 = 0\\
&u_2v_5(x_1y_1 - x_4y_4) + u_5v_2(x_4y_1 - x_1y_4) + u_2(v_4 - v_1)x_3y_2 = 0
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
&u_5v_2x_1y_2 - u_1v_7x_1y_3 + u_2v_5x_4y_2 - u_7v_4x_4y_3 + (u_2v_1 - u_1v_2)x_2y_1\\
&\qquad +(u_5v_7 - u_7v_5)x_3y_1 = 0\\
&u_5v_2x_1y_3 - u_1v_7x_1y_2 + u_2v_5x_4y_3 - u_7v_4x_4y_2 + (u_2v_1 - u_1v_2)x_3y_1\\
&\qquad +(u_5v_7 - u_7v_5)x_2y_1 = 0\\
&u_5v_2x_4y_2 - u_4v_7x_4y_3 + u_2v_5x_1y_2 - u_7v_1x_1y_3 + (u_2v_4 - u_4v_2)x_2y_4\\
&\qquad +(u_5v_7 - u_7v_5)x_3y_4 = 0\\
&u_5v_2x_4y_3 - u_4v_7x_4y_2 + u_2v_5x_1y_3 - u_7v_1x_1y_2 + (u_2v_4 - u_4v_2)x_3y_4\\
&\qquad +(u_5v_7 - u_7v_5)x_2y_4 = 0
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
&(u_1v_1 - u_2v_2)x_1y_2 + (u_7v_7 - u_5v_5)x_4y_2 - u_5v_1x_2y_1 + u_1v_5x_2y_4 - u_2v_7x_3y_1\\
&\qquad +u_7v_2x_3y_4 = 0\\
&(u_1v_1 - u_2v_2)x_1y_3 + (u_7v_7 - u_5v_5)x_4y_3 - u_5v_1x_3y_1 + u_1v_5x_3y_4 - u_2v_7x_2y_1\\
&\qquad +u_7v_2x_2y_4 = 0\\
&(u_4v_4 - u_2v_2)x_4y_2 + (u_7v_7 - u_5v_5)x_1y_2 - u_5v_4x_2y_4 + u_4v_5x_2y_1 - u_2v_7x_3y_4\\
&\qquad +u_7v_2x_3y_1 = 0\\
&(u_4v_4 - u_2v_2)x_4y_3 + (u_7v_7 - u_5v_5)x_1y_3 - u_5v_4x_3y_4 + u_4v_5x_3y_1 - u_2v_7x_2y_4\\
&\qquad +u_7v_2x_2y_1 = 0.
\end{aligned}
\tag{B.5}
$$

There also exist symmetries of interchanging indices. The system of equations (B) is invariant under the exchange of $a_1(u) \leftrightarrow a_4(u)$ and $\omega_1(u) \leftrightarrow \omega_4(u)$ or $a_2(u) \leftrightarrow a_3(u)$. The gauge R-matrix is [20],

$$
\begin{aligned}
\omega_1(u) &= \mathrm{cn}\,u + H\,\mathrm{sn}\,u\,\mathrm{dn}\,u\\
\omega_4(u) &= \mathrm{cn}\,u - H\,\mathrm{sn}\,u\,\mathrm{dn}\,u\\
\omega_2(u) &= \omega_3(u) = G\,\mathrm{sn}\,u\,\mathrm{dn}\,u\\
\omega_5(u) &= \omega_6(u) = \mathrm{dn}\,u\\
\omega_7(u) &= \omega_8(u) = k\,\mathrm{sn}\,u\,\mathrm{cn}\,u
\end{aligned}
\tag{2.10}
$$

where G, H are arbitrary parameters with relation $G^2 - H^2 = 1$. Note that in (2.10) we do not take $\omega_5(u) = \omega_6(u) = 1$ in order to compare our following discussion with other's work.

We will consider the general R-matrix which has $\omega_1(u) \neq \omega_4(u)$, i.e. $H \neq 0$. The case of $H = 0$ is remarked at the end of this section. Now we solve the RE (B) case by case.

Case 2.3.1: diagonal solution. From (B.1), $a_2(u) \equiv 0 \Leftrightarrow a_3(u) \equiv 0$. There are only two equations to be considered,

$$\begin{aligned} &u_2 v_5 (x_1 y_1 - x_4 y_4) + u_5 v_2 (x_4 y_1 - x_1 y_4) = 0 \\ &u_7 v_1 x_1 y_1 - u_1 v_7 x_1 y_4 + u_4 v_7 x_4 y_1 - u_7 v_4 x_4 y_4 = 0. \end{aligned} \tag{2.11}$$

Introducing a new variable $g(u) = a_1(u)/a_4(u)$ and solving $g(u)$ from the above equations, we get a solution

$$K_-(u) = \begin{pmatrix} \mathrm{cn}\, u\, \mathrm{dn}\, u \pm \mathrm{i}k' \,\mathrm{sn}\, u & 0 \\ 0 & \mathrm{cn}\, u\, \mathrm{dn}\, u \mp \mathrm{i}k' \,\mathrm{sn}\, u \end{pmatrix} \tag{2.12}$$

where k' is the complementary modulus of elliptic function. Note that the diagonal solution of 8V free-fermion type I has no free parameter, which is different from that of the Baxter type.

Case 2.3.2: skew-diagonal solution. If $a_2(u) \not\equiv 0$, we conclude from (B.1) that

$$a_2(u) = \epsilon a_3(u) \qquad \epsilon = \pm 1. \tag{2.13}$$

Taking $a_1(u) \equiv 0$, we get from (B.4)

$$x_4(u_2 v_5 y_2 - u_7 v_4 y_3) = 0.$$

With the help of (2.13) and (2.10), the above equation calls for $a_4(u) \equiv 0$. However, this is contradictory to $a_2(u) \not\equiv 0$ as seen from (B.2). So the skew-diagonal solution does not exist due to less symmetry of the R-matrix.

Case 2.3.3: general solution. Because $(u_1 - u_4)v_2 = (v_1 - v_4)u_2$, the following equation is obtained from (B.2), (B.3) and (2.13):

$$\begin{aligned} &u_7 v_1 x_1 y_1 - u_1 v_7 x_1 y_4 + u_4 v_7 x_4 y_1 - u_7 v_4 x_4 y_4 \\ &\qquad -\epsilon\{u_2 v_5 (x_1 y_1 - x_4 y_4) + u_5 v_2 (x_4 y_1 - x_1 y_4)\} = 0. \end{aligned} \tag{2.14}$$

Differentiating the above equation with respect to v and setting $v = 0$, we can express $a_1(u), a_4(u)$ as

$$\begin{aligned} a_1(u) &= (F(u)\, \mathrm{cn}\, u\, \mathrm{dn}\, u - G(u)\, \mathrm{sn}\, u) p(u)/2 \\ a_4(u) &= (F(u)\, \mathrm{cn}\, u\, \mathrm{dn}\, u + G(u)\, \mathrm{sn}\, u) p(u)/2 \end{aligned}$$

where

$$F(u) = c_1 + \frac{k((1 - \epsilon k G)c_1 + H c_2)}{\epsilon G - k} \mathrm{sn}^2\, u \tag{2.15}$$

$$E(u) = c_2 + \frac{k((1 - \epsilon k G)c_2 - k'^2 H c_1)}{\epsilon G - k} \mathrm{sn}^2\, u \tag{2.16}$$

and $p(u)$ is a meromorphic function to be determined. Substituting the above expressions into (B.5), we get

$$\frac{a_2(u)}{p(u)} = \mu\, \mathrm{sn}\, u\, \mathrm{cn}\, u\, \mathrm{dn}\, u$$

and an additional restriction between c_1 and c_2 from (B.2)

$$k'^2 c_1^2 + c_2^2 = 2\mu^2 (G - \epsilon k)/k.$$

Therefore, the most general solution is

$$K_-(u) = \begin{pmatrix} F(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u) + E(u)\,\mathrm{sn}(u) & 2\mu\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u) \\ 2\epsilon\,\mu\,\mathrm{sn}(u)\,\mathrm{cn}(u)\,\mathrm{dn}(u) & F(u)\,\mathrm{cn}(u)\mathrm{dn}(u) - E(u)\,\mathrm{sn}(u) \end{pmatrix}. \tag{2.17}$$

The result in [17] is a specific case of $\mu = 1$. It is also easy to find that the diagonal solution (2.12) can be obtained by setting $\mu = 0$.

Remark 2.3.1. *There are in fact various parametrizaions of a free-fermionic 8V R-matrix, one of which is given in [24]:*

$$\begin{aligned}
\omega_1(u) &= 1 - e(u)e(h_1)e(h_2) \\
\omega_4(u) &= e(u) - e(h_1)e(h_2) \\
\omega_2(u) &= e(h_2) - e(u)e(h_1) \\
\omega_3(u) &= e(h_1) - e(u)e(h_2) \\
\omega_5(u) &= \omega_6(u) = \sqrt{e(h_1)\,\mathrm{sn}(h_1)e(h_2)\,\mathrm{sn}(h_2)}(1 - e(u))/\,\mathrm{sn}\left(\frac{u}{2}\right) \\
\omega_7(u) &= \omega_8(u) = -\mathrm{i}k\sqrt{e(h_1)\,\mathrm{sn}(h_1)e(h_2)\,\mathrm{sn}(h_2)}(1 + e(u))\,\mathrm{sn}\left(\frac{u}{2}\right)
\end{aligned} \tag{2.18}$$

where h_1 and h_2 are colour parameters, and $e(u)$ is the elliptic exponential:

$$e(u) = \mathrm{cn}(u) + \mathrm{i}\,\mathrm{sn}(u).$$

If we make transformation (R.B) to (2.18) with factor function

$$\sqrt{e(h_1)e(h_2)\,\mathrm{sn}\,h_1\,\mathrm{sn}\,h_2}\,\frac{1 - e(u)}{\mathrm{sn}\,u/2}$$

and set

$$h_1 = h_2 = h, \qquad u \to u/2 \qquad G = \frac{1}{\mathrm{sn}\,h} \qquad H = \frac{\mathrm{cn}\,h}{\mathrm{sn}\,h}$$

the new R-matrix coincides with (2.10), so our solution includes the diagonal solution given in [16].

Remark 2.3.2. *Let us consider the special case of $H = 0$ and $G = 1$. In this case, the R-matrix is*

$$\begin{aligned}
\omega_1(u) &= \omega_4(u) = \mathrm{cn}\,u \\
\omega_2(u) &= \omega_3(u) = \mathrm{sn}\,u\,\mathrm{dn}\,u \\
\omega_5(u) &= \omega_6(u) = \mathrm{dn}\,u \\
\omega_7(u) &= \omega_8(u) = k\,\mathrm{sn}\,u\,\mathrm{cn}\,u.
\end{aligned} \tag{2.19}$$

It has the same symmetry as the Baxter type. The calculation shows that this feature is responsible for the fact that both two R-matrices share the same $K_-(u)$ as given in (2.9).

2.4. Free-fermion type II

This kind of R-matrix takes the form,

$$\begin{aligned}
\omega_1(u) &= \omega_4(u) = \frac{\cosh(\lambda u)}{\cos(\mu u)} \\
\omega_2(u) &= -\omega_3(u) = -\frac{\sinh(\lambda u)}{\cos(\mu u)} \\
\omega_5(u) &= \omega_6(u) = 1 \\
\omega_7(u) &= \omega_8(u) = \tan(\mu u)
\end{aligned} \tag{2.20}$$

where λ, μ are parameters. The RE in component forms is equivalent to the following 12 equations:

$$\begin{aligned}&u_2v_7(x_2y_2+x_3y_3)+u_5v_1(x_3y_2-x_2y_3)=0\\&u_7v_2(x_2y_2+x_3y_3)+u_1v_5(x_3y_2-x_2y_3)=0\end{aligned}\tag{C.1}$$

$$\begin{aligned}&u_7v_1(x_1y_1-x_4y_4)+u_1v_7(x_4y_1-x_1y_4)=0\\&u_2v_5(x_1y_1-x_4y_4)+u_5v_2(x_4y_1-x_1y_4)=0\end{aligned}\tag{C.2}$$

$$\begin{aligned}&u_5v_2x_1y_2-u_1v_7x_1y_3+u_2v_5x_4y_2-u_7v_1x_4y_3+(u_2v_1-u_1v_2)x_2y_1\\&\qquad+(u_5v_7-u_7v_5)x_3y_1=0\\&u_5v_2x_1y_3-u_1v_7x_1y_2+u_2v_5x_4y_3-u_7v_1x_4y_2+(u_2v_1-u_1v_2)x_3y_1\\&\qquad+(u_5v_7-u_7v_5)x_2y_1=0\\&u_5v_2x_4y_2-u_1v_7x_4y_3+u_2v_5x_1y_2-u_7v_1x_1y_3+(u_2v_1-u_1v_2)x_2y_4\\&\qquad+(u_5v_7-u_7v_5)x_3y_4=0\\&u_5v_2x_4y_3-u_1v_7x_4y_2+u_2v_5x_1y_3-u_7v_1x_1y_2+(u_2v_1-u_1v_2)x_3y_4\\&\qquad+(u_5v_7-u_7v_5)x_2y_4=0\end{aligned}\tag{C.3}$$

$$\begin{aligned}&(u_1v_1+u_2v_2)x_1y_2+(u_7v_7-u_5v_5)x_4y_2-u_5v_1x_2y_1+u_1v_5x_2y_4\\&\qquad+u_2v_7x_3y_1-u_7v_2x_3y_4=0\\&(u_1v_1+u_2v_2)x_1y_3+(u_7v_7-u_5v_5)x_4y_3-u_5v_1x_3y_1+u_1v_5x_3y_4\\&\qquad+u_2v_7x_2y_1-u_7v_2x_2y_4=0\\&(u_1v_1+u_2v_2)x_4y_2+(u_7v_7-u_5v_5)x_1y_2-u_5v_1x_2y_4+u_1v_5x_2y_1\\&\qquad+u_2v_7x_3y_4-u_7v_2x_3y_1=0\\&(u_1v_1+u_2v_2)x_4y_3+(u_7v_7-u_5v_5)x_1y_3-u_5v_1x_3y_4+u_1v_5x_3y_1\\&\qquad+u_2v_7x_2y_4-u_7v_2x_2y_1=0.\end{aligned}\tag{C.4}$$

We find that under the R-transformation of interchanging $\omega_2(u)$ and $\omega_3(u)$ in (2.20), one can perform a K-transformation as follows:

$$\bar{a}_2(u)=-a_2(u)\qquad\text{or}\qquad\bar{a}_3(u)=-a_3(u)\tag{2.21}$$

to keep the system of equations (C) invariant.

The existence of a non-trivial solution implies that there exists the relation

$$\frac{u_7v_2}{u_2v_7}=\frac{u_1v_5}{u_5v_1}$$

which requires $\lambda=\pm\mathrm{i}\mu$. Thus we should consider two different R-matrices,

$$\begin{aligned}&\omega_1(u)=\omega_4(u)=1\\&\omega_2(u)=-\omega_3(u)=\mathrm{i}\tan u\\&\omega_5(u)=\omega_6(u)=1\\&\omega_7(u)=\omega_8(u)=\tan u\end{aligned}\tag{2.22}$$

and

$$\begin{aligned}&\omega_1(u)=\omega_4(u)=1\\&\omega_2(u)=-\omega_3(u)=-\mathrm{i}\tan u\\&\omega_5(u)=\omega_6(u)=1\\&\omega_7(u)=\omega_8(u)=\tan u.\end{aligned}\tag{2.23}$$

They are in fact related each other by an exchange $\omega_2(u) \leftrightarrow \omega_3(u)$. Let us give solution $K_-(u)$ directly because the calculation procedure has nothing new. For $R(u)$ in (2.22), we have

$$K_-(u) = \begin{pmatrix} \mu_1(1+\nu_1\sin 2u) & \mathrm{i}\mu_2(1+\nu_2\cos 2u)\sin 2u \\ \mu_2(1-\nu_2\cos 2u)\sin 2u & \mu_1(1-\nu_1\sin 2u) \end{pmatrix} \tag{2.24}$$

while for $R(u)$ in (2.23), using the K-transformation (2.21), we have

$$K_-(u) = \begin{pmatrix} \mu_1(1+\nu_1\sin 2u) & \mathrm{i}\mu_2(1+\nu_2\cos 2u)\sin 2u \\ -\mu_2(1-\nu_2\cos 2u)\sin 2u & \mu_1(1-\nu_1\sin 2u) \end{pmatrix} \tag{2.25}$$

where $\mu_1, \mu_2, \nu_1, \nu_2$ are free parameters.

3. K_--matrix for the 6V model

The general 6V R-matrix takes the form

$$R(u) = \begin{pmatrix} \omega_1(u) & 0 & 0 & 0 \\ 0 & \omega_2(u) & \omega_5(u) & 0 \\ 0 & \omega_6(u) & \omega_3(u) & 0 \\ 0 & 0 & 0 & \omega_4(u) \end{pmatrix}. \tag{3.1}$$

By setting $u_{7,8} = 0$ and $v_{7,8} = 0$ in equations (A), we write down the reflection equations for the 6V-type R-matrix in component forms:

$$\begin{aligned} &(u_4 - u_1)v_2x_2y_2 = 0 \\ &(u_4 - u_1)v_3x_3y_3 = 0 \end{aligned} \tag{D.1}$$

$$\begin{aligned} &u_1v_5(x_2y_3 - x_3y_2) = 0 \\ &u_4v_6(x_2y_3 - x_3y_2) = 0 \end{aligned} \tag{D.2}$$

$$\begin{aligned} &(u_2v_3 - u_3v_2)x_1y_4 + u_5v_1(x_3y_2 - x_2y_3) = 0 \\ &(u_2v_3 - u_3v_2)x_4y_1 + u_6v_4(x_3y_2 - x_2y_3) = 0 \end{aligned} \tag{D.3}$$

$$\begin{aligned} &u_3v_6x_1y_1 - u_3v_5x_4y_4 - u_6v_3x_1y_4 + u_5v_3x_4y_1 + u_3(v_4 - v_1)x_3y_2 = 0 \\ &u_2v_6x_1y_1 - u_2v_5x_4y_4 - u_6v_2x_1y_4 + u_5v_2x_4y_1 + u_2(v_4 - v_1)x_2y_3 = 0 \end{aligned} \tag{D.4}$$

$$\begin{aligned} &u_6v_2x_1y_2 + u_2v_5x_4y_2 + (u_2v_1 - u_1v_2)x_2y_1 = 0 \\ &u_6v_3x_1y_3 + u_3v_5x_4y_3 + (u_3v_1 - u_1v_3)x_3y_1 = 0 \\ &u_5v_2x_4y_2 + u_2v_6x_1y_2 + (u_2v_4 - u_4v_2)x_2y_4 = 0 \\ &u_5v_3x_4y_3 + u_3v_6x_1y_3 + (u_3v_4 - u_4v_3)x_3y_4 = 0 \end{aligned} \tag{D.5}$$

$$\begin{aligned} &(u_1v_1 - u_3v_2)x_1y_2 - u_5v_5x_4y_2 - u_5v_1x_2y_1 + u_1v_5x_2y_4 = 0 \\ &(u_1v_1 - u_2v_3)x_1y_3 - u_5v_5x_4y_3 - u_5v_1x_3y_1 + u_1v_5x_3y_4 = 0 \\ &(u_4v_4 - u_3v_2)x_4y_2 - u_6v_6x_1y_2 - u_6v_4x_2y_4 + u_4v_6x_2y_1 = 0 \\ &(u_4v_4 - u_2v_3)x_4y_3 - u_6v_6x_1y_3 - u_6v_4x_3y_4 + u_4v_6x_3y_1 = 0. \end{aligned} \tag{D.6}$$

From [21], we know that the 6V-type solutions of YBE have the same solution-transformation as that for 8V-type solutions except for the symmetries of weight functions and of the interchanging indices related to $\omega_7(u)$ and $\omega_8(u)$. Now the two symmetries of weight functions are

$$\bar{\omega}_2(u) = s\omega_2(u) \qquad \bar{\omega}_3(u) = s^{-1}\omega_3(u) \tag{3.2}$$

and

$$\bar{\omega}_5(u) = e^{cu}\omega_5(u) \qquad \bar{\omega}_6(u) = e^{-cu}\omega_6(u) \tag{3.3}$$

where s, c are two non-zero constants. In fact, we find that transformation (3.2) has no effect on the system of equations (D), and if making K-transformation

$$\bar{a}_1(u) = e^{cu}a_1(u) \qquad \bar{a}_4(u) = e^{-cu}a_4(u) \tag{3.4}$$

the new $K_-(u)$ is still a solution to RE for the new R-matrix obtained from R-transformation (3.3). Due to these symmetries, we will consider the gauge R-matrices as in the 8V model. They are also classified into two classes: the Baxter type

$$\begin{aligned} &\omega_1(u) = \omega_4(u) = \frac{\sin(u+h)}{\sin h} \\ &\omega_2(u) = \omega_3(u) = \frac{\sin u}{\sin h} \\ &\omega_5(u) = \omega_6(u) = 1 \end{aligned} \tag{3.5}$$

and the free-fermion type

$$\begin{aligned} &\omega_1(u) = \frac{\sin(u+h)}{\sin h} \\ &\omega_4(u) = \frac{\sin(-u+h)}{\sin h} \\ &\omega_2(u) = \omega_3(u) = \frac{\sin u}{\sin h} \\ &\omega_5(u) = \omega_6(u) = 1. \end{aligned} \tag{3.6}$$

For the Baxter type, the general solution to RE was given in [13]

$$K_-(u) = \begin{pmatrix} \lambda\sin(\alpha-u) & \mu\sin(2u) \\ \nu\sin(2u) & \lambda\sin(\alpha+u) \end{pmatrix} \tag{3.7}$$

which has four free parameters λ, α, μ and ν.

While for the free-fermion type, since $\omega_1(u) \neq \omega_4(u)$, one can immediately see that $a_2(u) \equiv 0$ and $a_3(u) \equiv 0$ from (D.1). In other words, the RE for the free-fermion-type 6V model only has the diagonal solution,

$$K_-(u) = \begin{pmatrix} \sin(\alpha-u) & 0 \\ 0 & \sin(\alpha+u) \end{pmatrix}. \tag{3.8}$$

In addition, if setting $\cos h = 0$ in (3.6), we have a symmetric R-matrix of free-fermion type as follows,

$$R(u) = \begin{pmatrix} \cos u & 0 & 0 & 0 \\ 0 & \sin u & 1 & 0 \\ 0 & 1 & \sin u & 0 \\ 0 & 0 & 0 & \cos u \end{pmatrix}. \tag{3.9}$$

Just as discussed in remark 2.3.2, this R-matrix shares the same K_--matrix in (3.7) with the 6V Baxter type.

So, up to K-transformation (3.4), we obtain all general solutions (3.7) and (3.8) to RE in the 6V case.

4. K_--matrices to the 7V model

If setting $\omega_8(u) \equiv 0$ in the 8V R-matrix (2.1), we get the 7V one

$$R(u) = \begin{pmatrix} \omega_1(u) & 0 & 0 & \omega_7(u) \\ 0 & \omega_2(u) & \omega_5(u) & 0 \\ 0 & \omega_6(u) & \omega_3(u) & 0 \\ 0 & 0 & 0 & \omega_4(u) \end{pmatrix}. \tag{4.1}$$

The classification of solutions to the coloured 7V-type YBE was given recently in [26]. Due to less symmetries, the R-matrices show a much more different properties from that of both 8V and 6V models. At this point, we expect that the corresponding RE reveals new features as well.

First of all, let us study the symmetries of the RE as was done previously for other cases. After removing the terms containing u_8 and v_8 in the system of equations (A), we find that there still exists K-transformation (2.4) under R-transformation (2.3) (note that ω_8 is absent!).

In [26], an additional relation $\omega_5(u)/\omega_6(u) = \mathrm{e}^{cu}$ is given, where c is a constant. When $c \neq 0$, there have only trivial K_--matrices. The case of $c = 0$ or $\omega_5(u) = \omega_6(u)$ is further classified into three different types: Baxter type, free-fermion type I and II, which will be discussed in the following sections.

4.1. Baxter type

The parametrization of the R-matrix is as follows:

$$\begin{aligned} \omega_1(u) &= \omega_4(u) = \frac{\sin(u+h)}{\sin h} \\ \omega_2(u) &= \omega_3(u) = \frac{\sin u}{\sin h} \\ \omega_5(u) &= \omega_6(u) = 1 \\ \omega_7(u) &= \sin(u+h)\sin u. \end{aligned} \tag{4.2}$$

Substituting (4.2) into system (A), we solve these equations case by case.

Case 4.1.1: diagonal solution. It can be seen from (A.1) that if $a_2(u) \equiv 0$ then $a_3(u) \equiv 0$. (Note that $u_8 = 0 = v_8$ in (A).) We obtain the diagonal solution:

$$K_-(u) = \begin{pmatrix} \sin(\alpha - u) & 0 \\ 0 & \sin(\alpha + u) \end{pmatrix}. \tag{4.3}$$

Case 4.1.2: skew-diagonal solution. Let $a_1(u) \equiv 0$, it requires $a_4(u) \equiv 0$ from (A.4). We only need to consider two equations:

$$\begin{aligned} u_7 v_3 x_3 y_3 + u_1 v_5 (x_2 y_3 - x_3 y_2) &= 0 \\ u_3 v_7 x_3 y_3 + u_5 v_1 (x_2 y_3 - x_3 y_2) &= 0. \end{aligned} \tag{4.4}$$

Solving equations (4.4), we have two K_--matrices,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \rho(u) \\ 1 & 0 \end{pmatrix}$$

where $\rho(u) = (\lambda + \cos 2u)/2$ and λ is a free parameter.

Case 4.1.3: $a_3(u) \equiv 0$. One can get from case 4.1.1 that

$$a_1(u) = p(u)\sin(\alpha - u) \qquad a_4(u) = p(u)\sin(\alpha + u)$$

and find

$$a_2(u)/p(u) = \mu \sin(2u)$$

so the K_--matrix is

$$K_-(u) = \begin{pmatrix} \nu\sin(\alpha - u) & \mu\sin(2u) \\ 0 & \nu\sin(\alpha + u) \end{pmatrix} \tag{4.5}$$

where μ, ν are parameters.

Case 4.1.4. Combining the results obtained above, one can easily write the general K-matrix as follows:

$$K_-(u) = \begin{pmatrix} \nu\sin(\alpha - u) & \mu\rho(u)\sin(2u) \\ \mu\sin(2u) & \nu\sin(\alpha + u) \end{pmatrix}. \tag{4.6}$$

In summary, we can regard (4.5) and (4.6) as the most general reflection matrices, because others can be obtained by assigning special values to free parameters. Furthermore, comparing (4.3) and (4.5), we see that in the case of the 7V type, $a_2(u) = 0$ implies $a_3(u) = 0$, but the reverse does not hold, this is different from the case of the 8V type.

4.2. Free-fermion type I

This kind of R-matrix reads

$$\begin{aligned} \omega_1(u) &= \frac{\sin(u+h)}{\sin h} \\ \omega_4(u) &= \frac{\sin(-u+h)}{\sin h} \\ \omega_2(u) &= \omega_3(u) = \frac{\sin u}{\sin h} \\ \omega_5(u) &= \omega_6(u) = 1 \\ \omega_7(u) &= \frac{\sin 2u}{\sin h}. \end{aligned} \tag{4.7}$$

If $\cos h = 0$, or $\omega_1(u) = \omega_4(u)$, we can make the similar calculation as for the Baxter type, as both have the same symmetries. The result is almost the same as that given in (4.6) and (4.5) but with $\rho(u) = (\lambda + \cos 2u)$.

When $\omega_1(u) \neq \omega_4(u)$, it forces $a_3(u) \equiv 0$ from the second equation of (A.3). The RE reduces to the following equations:

$$\begin{aligned} &u_2v_5(x_1y_1 - x_4y_4) + u_5v_2(x_4y_1 - x_1y_4) = 0 \\ &u_7v_1x_1y_1 - u_7v_1x_4y_4 + u_4v_7x_4y_1 - u_1v_7x_1y_4 + (u_4 - u_1)v_2x_2y_2 = 0 \end{aligned} \tag{E.1}$$

$$\begin{aligned} &u_5v_2x_1y_2 + u_2v_5x_4y_2 + (u_2v_1 - u_1v_2)x_2y_1 = 0 \\ &u_5v_2x_4y_2 + u_2v_5x_1y_2 + (u_2v_4 - u_4v_2)x_2y_4 = 0 \\ &(u_1v_1 - u_3v_2)x_1y_2 - u_5v_5x_4y_2 - u_5v_1x_2y_1 + u_1v_5x_2y_4 = 0 \\ &(u_4v_4 - u_3v_2)x_4y_2 - u_5v_5x_1y_2 - u_5v_4x_2y_4 + u_4v_5x_2y_1 = 0. \end{aligned} \tag{E.2}$$

Case 4.2.1: $a_2(u) \equiv 0$. The two equations of (E.1) are not compatible with each other due to the symmetry between $\omega_1(u)$ and $\omega_4(u)$ being broken. Therefore there exists no diagonal solution for this type of R-matrix.

Case 4.2.2: $a_1(u) \equiv 0$. One can deduce from (E.1) that $a_4(u) \equiv 0$ and $a_2(u) \equiv 0$. This is a trivial case.

Case 4.2.3: General solution. From the second equation of (E.1) and (E.2), one can get the same result as that in the Baxter type

$$a_1(u) = \sin(\alpha - u) \qquad a_4(u) = \sin(\alpha + u) \qquad a_2(u) = \mu \sin 2u.$$

Substituting them into the second equation of (E.1), one finds $\mu = \pm 1$. So, we have only one solution

$$K_-(u) = \begin{pmatrix} \sin(\alpha - u) & \pm \sin(2u) \\ 0 & \sin(\alpha + u) \end{pmatrix} \tag{4.8}$$

which also shows that $a_3(u) = 0$ does not imply $a_2(u) = 0$.

4.3. Free-fermion type II

In this case, the elements of the R-matrix take the following forms:

$$\begin{aligned} &\omega_1(u) = \omega_4(u) = \cosh u \\ &\omega_2(u) = -\omega_3(u) = \sinh u \\ &\omega_5(u) = \omega_6(u) = 1 \\ &\omega_7(u) = u. \end{aligned} \tag{4.9}$$

For the sake of brevity, we simply give the result. Note that the non-triviality requires $a_1(u) = \pm a_4(u)$. If $a_1(u) = a_4(u)$, we get

$$K_-(u) = \begin{pmatrix} \alpha & \mu \sinh u \\ 0 & \alpha \end{pmatrix} \tag{4.10}$$

while if $a_1(u) = -a_4(u)$, we have

$$K_-(u) = \begin{pmatrix} \alpha & \mu \cosh u \\ 0 & -\alpha \end{pmatrix}. \tag{4.11}$$

In addition, according to the discussion in section 2.4, the solutions remain invariant under exchange of $\omega_2 \leftrightarrow \omega_3$ since $a_3(u) \equiv 0$.

5. Construction of boundary Hamiltonians

In this section, we will discuss the Hamiltonians for the systems described by the R-matrices and K-matrices obtained in the previous sections. The systems with such Hamiltonians are open integrable quantum spin-chains. The 6V (Baxter type and free-fermion type) and 8V (Baxter type and free-fermion type I) are included in Sklyanin's formalism. While for 7V models, both Baxter type and free-fermion type-I R-matrices have only regularity, P-symmetry, unitarity and crossing-unitarity symmetries. To assure that the transfer matrix $t(u)$ in these cases commutes to each other for different spectrum parameters, the RE for the boundary matrix $K_+(u)$ is generalized from equation (1.3) to (1.5), so the corresponding $K_+(u)$-matrices may be obtained by equation (1.6). The integrability of the systems constructed from the above transfer matrix is guaranteed (see [12]). Considering that all of these cases have the same definition of transfer matrix [8,12], we can construct their Hamiltonians in a unified way.

If $K_-(0) \propto \mathrm{id}$, $\mathrm{tr}\, K_+(0) \neq 0$, the Hamiltonian for the open systems, which is obtained from the first derivative of the transfer matrix, is defined as [8]

$$\mathcal{H} \equiv \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} + \frac{1}{2} K_-^{-1}(0) \overset{1}{K'_-}(0) + \frac{\mathrm{tr}_0 \overset{0}{K_+}(0)\mathcal{H}_{N0}}{\mathrm{tr}\, K_+(0)} \tag{5.1}$$

where two-site Hamiltonian $\mathcal{H}_{j,j+1}$ is given by

$$\mathcal{H}_{j,j+1} = P_{j,j+1} \frac{\mathrm{d}}{\mathrm{d}u} R_{j,j+1}(u)|_{u=0} = \frac{\mathrm{d}}{\mathrm{d}u} R_{j,j+1}(u)|_{u=0} P_{j,j+1}. \tag{5.2}$$

All the Baxter-type models in two-component systems belong to this case. The boundary Hamiltonian of the 6V- and 8V-Baxter type can be found in [13, 15, 18].

Case 5.1: Baxter-type 7V with crossing parameter $\eta = h$. From $K_-(u)$ in (4.6) and the relation (1.6), we find that

$$K_+(u) = K_-(-u - h; -\alpha_+, \mu_+, \nu_+, \lambda_+). \tag{5.3}$$

According to equation (5.1), the Hamiltonian is

$$\mathcal{H} = \frac{1}{4\sin h} \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} - A_-\sigma_1^z + B_-\sigma_1^+ + C_-\sigma_1^- - A_+\sigma_N^z + B_+\sigma_N^+ + C_+\sigma_N^- \tag{5.4}$$

where

$$\begin{aligned}\mathcal{H}_{j,j+1} = {} & (2 + \sin^2 h)\sigma_j^x\sigma_{j+1}^x + (2 - \sin^2 h)\sigma_j^y\sigma_{j+1}^y + \mathrm{i}\sin^2 h(\sigma_j^x\sigma_{j+1}^y + \sigma_j^y\sigma_{j+1}^x) \\ & +2\cos h\sigma_j^z\sigma_{j+1}^z\end{aligned} \tag{5.5}$$

$$A_\pm = \frac{1}{2}\cot\alpha_\pm \qquad B_\pm = \frac{(1+\lambda_\pm)\mu_\pm}{2\sin\alpha_\pm} \qquad C_\pm = \frac{\mu_\pm}{\nu_\pm \sin\alpha_\pm}. \tag{5.6}$$

However, if $\mathrm{tr}\, K_+(0) = 0$, as pointed out in [13, 16], there will be no well-defined Hamiltonian from the first derivative of the transfer matrix as in (5.1). But if

$$\mathrm{tr}_0 \overset{0}{K_+}(0)\mathcal{H}_{N0} = A \cdot \mathrm{id} \tag{5.7}$$

where A is a constant, we can still derive the well-defined local Hamiltonian from the second derivative of transfer matrix as follows:

$$\begin{aligned}\mathcal{H} \equiv \frac{t''(0)}{4(C+2A)} = {} & \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} + \frac{1}{2} K_-^{-1}(0) \overset{1}{K'_-}(0) \\ & + \frac{1}{2(C+2A)}\{\mathrm{tr}_0(\overset{0}{K_+}(0)\mathcal{G}_{N0}) + 2\,\mathrm{tr}_0(\overset{0}{K'_+}(0)\mathcal{H}_{N0}) + \mathrm{tr}_0(\overset{0}{K_+}(0)\mathcal{H}_{N0}^2)\}\end{aligned} \tag{5.8}$$

where

$$C \equiv \mathrm{tr}\, K'_+(0) \tag{5.9}$$

$$\mathcal{G}_{j,j+1} \equiv P_{j,j+1} \left.\frac{\mathrm{d}^2 R_{j,j+1}(u)}{\mathrm{d}u^2}\right|_{u=0}. \tag{5.10}$$

The following discussions show that all the boundary conditions corresponding to the free-fermion type R-matrix belong to this case. We argue that it is a common property for all free-fermion models.

Case 5.2: free-fermion type-I 8V with crossing parameter $\eta = I$. Here I is the complete elliptic integral of the first kind of modulus k. For general boundary condition described by $K_-(u)$ in (2.17), we have

$$K_+(u) = K_-^t(-u - I)$$
$$= \begin{pmatrix} k'^2 F_+(u)\,\mathrm{sn}\,u + E_+(u)\,\mathrm{cn}\,u\,\mathrm{dn}\,u & 2\mu_+ k'^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u \\ 2\epsilon\mu_+ k'^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u & k'^2 F_+(u)\mathrm{sn}u - E_+(u)\,\mathrm{cn}\,u\,\mathrm{dn}\,u \end{pmatrix} \quad (5.11)$$

where

$$F_+(u) = c_1^+\,\mathrm{dn}^2\,u + \frac{k((1-\epsilon kG)c_1^+ + Hc_2^+)}{\epsilon G - k}\,\mathrm{cn}^2\,u \quad (5.12)$$

$$E_+(u) = c_2^+\,\mathrm{dn}^2\,u + \frac{k((1-\epsilon kG)c_2^+ - k'^2 Hc_1^+)}{\epsilon G - k}\,\mathrm{cn}^2\,u. \quad (5.13)$$

From equation (5.8), the Hamiltonian reads

$$\mathcal{H} = \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} + A_-\sigma_1^z + B_-(\sigma_1^+ + \epsilon\sigma_1^-) + A_+\sigma_N^z + B_+(\sigma_N^+ + \epsilon\sigma_N^-) \quad (5.14)$$

where

$$\mathcal{H}_{j,j+1} = \frac{H}{2}(\sigma_j^z + \sigma_{j+1}^z) + \frac{G+k}{2}\sigma_j^x\sigma_{j+1}^x + \frac{G-k}{2}\sigma_j^y\sigma_{j+1}^y \quad (5.15)$$

and

$$A_- = c_2^-/c_1^- \qquad B_- = 2\mu^-/c_1^-$$
$$A_+ = k'^2(HF_+(0) - E_+(0))/2(k'^2 F_+(0) + HE_+(0))$$
$$B_+ = k'^2(G+\epsilon k)\mu^+/(k'^2 F_+(0) + HE_+(0)).$$

For the diagonal K-matrix (2.12), we have

$$\mathcal{H} = \sum_{j=1}^{N-1} \mathcal{H}_{j,j+1} + \frac{\mathrm{i}k'}{2}(\sigma_1^z - \sigma_N^z) \quad (5.16)$$

which has been discussed in [16] and is a special case of (5.14).

Case 5.3: symmetric free-fermion type-I 8V. If considering R-matrix (2.19) and the corresponding K-matrix (2.9), we get the following Hamiltonian:

$$\mathcal{H} = \sum_{j=1}^{N-1}\left(\frac{1+k}{2}\sigma_j^x\sigma_{j+1}^x + \frac{1-k}{2}\sigma_j^y\sigma_{j+1}^y\right) - A_-\sigma_1^z + B_-\sigma_1^+ + C_-\sigma_1^- - A_+\sigma_N^z + B_+\sigma_N^+ + C_+\sigma_N^- \quad (5.17)$$

where

$$A_- = \frac{\mathrm{cn}\,\alpha_\pm\,\mathrm{dn}\,\alpha_\pm}{2\,\mathrm{sn}\,\alpha_\pm} \qquad A_+ = \frac{\mathrm{cn}\,\alpha_+}{2\,\mathrm{sn}\,\alpha_+\,\mathrm{dn}\,\alpha_+}(1 - k^2\,\mathrm{sn}\,\alpha_+)$$
$$B_\pm = \frac{\mu_\pm(\lambda_\pm + 1)}{\mathrm{sn}\,\alpha_\pm} \qquad C_\pm = \frac{\mu_\pm(\lambda_\pm - 1)}{\mathrm{sn}\,\alpha_\pm}.$$

Case 5.4: free-fermion-type 6V with $\eta = \pi/2$. For the R-matrix in (3.6) and the general K_--matrix (3.8), if setting

$$K_+(u) = K_-^t(-u - \pi/2; \pi/2 - \alpha_+ - h) \quad (5.18)$$

we have

$$\mathcal{H} = \frac{1}{\sin h}\sum_{j=1}^{N-1}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cos h(\sigma_j^z + \sigma_{j+1}^z)) - \cot\alpha_-\sigma_1^z - \cot\alpha_+\sigma_N^z. \quad (5.19)$$

Case 5.5: symmetric free-fermion-type 6V. If considering $R(u)$ in (3.9) together with the general K_--matrix (3.7), we can set

$$K_+(u) = K_-^t(-u - \pi/2; -\alpha_+, \mu_+, \nu_+) \tag{5.20}$$

thus the Hamiltonian is

$$\mathcal{H} = \sum_{j=1}^{N-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) - A_- \sigma_1^z + B_- \sigma_1^+ + C_- \sigma_1^- - A_+ \sigma_N^z + B_+ \sigma_N^+ + C_+ \sigma_N^- \tag{5.21}$$

where

$$A_\pm = \cot \alpha_\pm \qquad B_\pm = \frac{2\mu_\pm}{\sin \alpha_\pm} \qquad C_\pm = \frac{2\nu_\pm}{\sin \alpha_\pm}.$$

Case 5.6: free-fermion type-I 7V with crossing parameter $\eta = \pi/2$. From the K_--matrix in (4.8) and the relation (1.6), we get

$$K_+(u) = K_-(-u - \pi/2; \alpha_+ - h + \pi/2, \mu_+) \tag{5.22}$$

and the Hamiltonian is

$$\begin{aligned} \mathcal{H} = \frac{1}{2\sin h} \sum_{j=1}^{N-1} \{\cos h(\sigma_j^z + \sigma_{j+1}^z) + 2\sigma_j^x \sigma_{j+1}^x + \mathrm{i}(\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x)\} - A_- \sigma_1^z + B_- \sigma_1^+ \\ - A_+ \sigma_N^z + B_+ \sigma_N^+ \end{aligned} \tag{5.23}$$

where

$$\mu_-, \mu_+ = \pm 1 \qquad A_\pm = \cot \alpha_\pm/2 \qquad B_\pm = \frac{\mu_\pm}{\sin \alpha_\pm}.$$

Case 5.7: symmetric free-fermion type-I 7V. From K_--matrix in (4.6) with $\rho(u) = \lambda + \cos 2u$ and the relation (1.6), we have

$$K_+(u) = K_-(-u - \pi/2; -\alpha_+, \mu_+, \nu_+, \lambda_+) \tag{5.24}$$

and

$$\begin{aligned} \mathcal{H} = \sum_{j=1}^{N-1} \left\{ \sigma_j^x \sigma_{j+1}^x + \frac{\mathrm{i}}{2}(\sigma_j^x \sigma_{j+1}^y + \sigma_j^y \sigma_{j+1}^x) \right\} - A_- \sigma_1^z + B_- \sigma_1^+ + C_- \sigma_1^- \\ - A_+ \sigma_N^z + B_+ \sigma_N^+ + C_+ \sigma_N^- \end{aligned} \tag{5.25}$$

where

$$A_\pm = \frac{\cot \alpha_\pm}{2} \qquad B_\pm = \frac{(1 + \lambda_\pm)\mu_\pm}{\nu_\pm \sin \alpha_\pm} \qquad C_\pm = \frac{\mu_\pm}{\nu_\pm \sin \alpha_\pm}.$$

It should be pointed out that for the free-fermion type II of both 7V and 8V models which have no crossing-unitarity symmetry, how to prove their integrability and to obtain the corresponding Hamiltonians in the case of open boundary condition is still an open problem.

6. Remarks and discussions

In this paper we found that symmetries play an important role in solving the reflection equation. For any non-standard R-matrix which is obtained by applying R-transformation to the standard one, the corresponding reflection matrix can then be obtained by making a K-transformation to that for the standard R-matrix.

Moreover, all solutions given above indicate that the number of free parameters appearing in a K_--matrix is determined by the symmetries of the R-matrix. R-matrices with different forms but the same symmetries share the same $K_-(u)$ matrix. The free-fermion-type R-matrix with $\omega_1(u) = \omega_4(u)$ is just that in this case. It has the same form $K_-(u)$ as in the Baxter type. Also we note that, different from that for 6V and 8V cases, the elements $a_2(u)$, $a_3(u)$ of $K_-(u)$ in the 7V case have no interchanging symmetry resulting from the symmetry between $\omega_7(u)$ and $\omega_8(u)$ of the R-matrix being broken.

It is also interesting to note that while constructing the Hamiltonian, all reflection matrices for free-fermion R-matrices have the property of tr $K_+(0) = 0$. We argue that it is a typical property for all free-fermion models. So the local Hamiltonian for such systems are obtained from the second derivative of the transfer matrix.

We are sure that our procedure to find solutions of the reflection equation can be applied to high-spin models, though the calculation may be much more involved in this case. With the solutions given in this paper, we can use the Bethe ansatz method to study the physical properties of open spin-chains.

Furthermore, much attention has been recently directed to the Yang–Baxter equation with dynamical parameters [27, 28]. How to construct the corresponding reflection equation and to seek its solution is an open problem. We wish to discuss some related problems using the method and procedure given in this paper.

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